Exercise 4

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$1 + \frac{1}{3}x^3 + xe^x - e^x = \int_0^x tu(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

Substitute this and the Taylor series expansion of e^x,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots ,$$

into the integral equation.

$$\begin{aligned} 1 + \frac{1}{3}x^3 + x\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right) \\ - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right) \\ = \int_0^x t(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots)\,dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} 1 + x + x^2 + \frac{5x^3}{6} + \frac{x^4}{6} + \frac{x^5}{24} - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{120} + \cdots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4 + \frac{a_3}{5}x^5 + \cdots \end{aligned}$$

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots = \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4 + \frac{a_3}{5}x^5 + \cdots$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{2} & \rightarrow \quad & a_0 = 1 \\ \frac{a_1}{3} &= \frac{2}{3} & \rightarrow \quad & a_1 = 2 \\ \frac{a_2}{4} &= \frac{1}{8} & \rightarrow \quad & a_2 = \frac{1}{2} \\ \frac{a_3}{5} &= \frac{1}{30} & \rightarrow \quad & a_3 = \frac{1}{6} \\ &\vdots & & \vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ &= x + e^x. \end{aligned}$$